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LOCALIZED AND VOLUME INTERNAL WAVES IN A STRATIFIED FLUID CONTIGUOUS TO A MIXED LAYER†

Yu. V. KISTOVICH and Yu. D. CHASHECHKIN

Moscow

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Dispersion relations, the rate of energy transfer, orthogonality and completeness relations and, also, the functions describing the vertical structure of two types of internal waves (perturbations localized close to the boundary and volume perturbations), which exist in an exponentially stratified medium contiguous to a homogeneous layer of finite thickness without any discontinuity in the density are calculated without recourse to the Boussinesq approximation. © 1998 Elsevier Science Ltd. All rights reserved.

The general theory of waves in inhomogeneous media indicates the need for a more detailed description of the permissible periodic motions, taking account of the possibility of the simultaneous existence of both surface and volume modes (as an example, one may mention acoustic Rayleigh waves [1], surface electromagnetic waves [2] and the Tamm (surface) states of quantum mechanical particles [3]). A general approach [4] to the simultaneous description of volume and surface waves was proposed in [4] and an analysis of the orthogonality and completeness relations was presented. A more detailed analysis of the permissible forms of internal waves in a non-uniformly stratified medium, which models the typical states of natural systems (of the ocean and atmosphere), when there is a layer of a homogeneous fluid under or above a fluid of variable density without any discontinuity in the density at the boundary, is of interest. The analysis of the natural forms of motion of such a medium enable one to find the complete orthonormalized system of eigenfunctions (wave modes) and the corresponding dispersion relations which can be used to study the evolution and decay of arbitrary initial perturbations or to solve problems with sources [5].

The density distribution scheme considered here includes a layer of an ideal incompressible fluid of thickness a which is bounded below by a solid base at $z = 0$. On top of this layer, there is an exponentially stratified fluid which is unrestricted in height and has a buoyancy frequency $N = \sqrt{(g/\Lambda)}$, where g is the acceleration due to gravity, which is directed opposite to the z axis. The unperturbed density distribution throughout the whole depth of the fluid can be expressed by the formula

$$\rho_0(z)=\begin{cases}\rho_{00}, & 0<z<a\\ \rho_{00}e^{-(z-a)/\Lambda}, & z>a\end{cases} \qquad (1)$$

that is, the fluid density is continuous in the boundary of the layer and Λ is the stratification scale. Such a density distribution is typical in the case of the atmospheres of planets and stars.

A monochromatic perturbation with a frequency ω, which is characterized by a variable velocity (υ_x, υ_z), density ρ and pressure P, is described in the linear approximation by the equations

$$i\omega\rho_0(z)\upsilon_x = -\partial P/\partial x, \ -i\omega\rho_0(z)\upsilon_z = -\partial P/\partial z - \rho g$$
$$-i\omega\rho + \upsilon_z d\rho_0/dz = 0, \ \partial\upsilon_x/\partial x + \partial\upsilon_z/\partial z = 0$$

On eliminating all of the variables here apart from υ_z and introducing the variable

$$u=\begin{cases}\upsilon_z, & 0<z<a\\ \upsilon_z e^{-\mu(z-a)}, & z>a, \ \mu = 1/(2\Lambda)\end{cases} \qquad (2)$$

we obtain, when not invoking the Boussinesq approximation, the equation

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$$(1-\zeta^2)\partial^2 u/\partial x^2+\partial^2 u/\partial z^2=\mu^2 u,\ \zeta^2=N^2/\omega^2 \tag{3}$$

The following mathematical formulation of the problem can therefore be given: it is required to find a function $u(x, z)$ which satisfies the Laplace equation when $0 < z < a$ and, when $z > a$, satisfies Eq. (3) with the boundary condition $u(x, 0) = 0$ and is bounded when $z \to \infty$ such that the function $\upsilon_z(x, z) = \{u(x, z)$ when $0 < z < a$ and $u(x, z)e^{\mu(z-a)}$ is continuously differentiable in the domain $z > 0$, $-\infty < x < +\infty$ when $a < z\ \infty\}$.

In accordance with the general approach (the method of separation of the variables), the solution of the problem is sought in the form of a wave perturbation localized close to the boundary of the homogeneous layer (the analogue of waves at an interface which decay exponentially with distance from it)

$$u=\begin{cases}(Ae^{i\lambda_0 z}+Be^{-i\lambda_0 z})e^{ikx}, & 0<z<a\\ Ce^{i\lambda(z-a)}e^{ikx}, & z>0,\ \mathrm{Im}\,\lambda>0\end{cases} \tag{4}$$

as well as in the form of the usual internal waves [5] which propagate in the stratified fluid and are reflected from the homogeneous layer

$$u=\begin{cases}(A_1e^{i\lambda_1 z}+B_1e^{-i\lambda_1 z})e^{ik_p x}, & 0<z<a\\ C_1[e^{-ip(z-a)}+\alpha(p)e^{ip(z-a)}]e^{ik_p x}, & z>0,\ \mathrm{Im}\,p=0\end{cases} \tag{5}$$

where $\alpha(p)$ is the reflection coefficient.

If the thickness of the homogeneous layer tends to zero, these waves become the well-known travelling internal waves. Their properties, including the geometry of reflection from a rigid plane boundary (taking account of dissipative effects), have been studied in detail in [6]. We note that the upper lines in (4) and (5) are particular solutions of Laplace's equation while the lower lines are particular solutions of Eq. (3). Only the localized waves (4) will be considered in detail below. In the case of volume internal waves, only the final results will be formulated.

Substituting the particular solution (4) into Eq. (3) and equating the determinant of the resulting homogeneous system of linear equations in the amplitudes to zero, we obtain two dispersion equations which relate λ_0, λ and k

$$k^2+\lambda_0^2=0,\ (1-\zeta^2)k^2+\lambda^2=-\mu^2 \tag{6}$$

Expressing the velocity components υ_x and υ_z using (4) and taking account of their continuity when $z = a$ and also the fact that the component υ_z is equal to zero when $z = 0$, we obtain

$$i\lambda_0(e^{i\lambda_0 a}+e^{-i\lambda_0 a})=(\mu+i\lambda)(e^{i\lambda_0 a}-e^{-i\lambda_0 a}) \tag{7}$$

To be specific, we shall consider the localized wave perturbations which propagate to the right along the x axis ($k > 0$) and then, from (6), we have

$$\lambda_0=ik,\ \lambda=i\mu_*(k),\ \mu_*(k)=\sqrt{\mu^2+(1-\zeta^2)k^2} \tag{8}$$

which, after substitution into (7), gives

$$\mu-\mu_*(k)=k\,\mathrm{cth}\,ka \tag{9}$$

whereupon the relation between ω and k is established, that is, the dispersion relation for the localized waves

$$\omega^2(k)=gk/(\mathrm{cth}\,ka-k\Lambda/\mathrm{sh}^2\,ka) \tag{10}$$

The right-hand side of Eq. (10) is an even function of k and the assumption that k is positive imposes no restriction on the generality of the subsequent treatment.

Localized perturbations with a dispersion (10) do not exist for all k as, since $\omega^2 > 0$ and k cth $ka > 0$, the left-hand side of Eq. (9) and the expression in parentheses in (10) must be positive. These conditions are equivalent to the inequalities $a > \Lambda$, $k < k_c$, where k_c is a root of the equation $k_c\Lambda$ cth $k_c a = 1$ and, as follows from (10), $\omega(k_c) = N$.

The condition for a non-negative value of the expression under the square root sign in (8) is automatically satisfied since, by virtue of (10), it is equivalent to the inequality $(\mu - k \operatorname{cth} ka)^2 \geqslant 0$.

For small k, the dispersion relation takes the form

$$\omega = kv, \quad v = a\sqrt{g/(a-\Lambda)}$$

when the phase and group velocities are independent of the wavelength, while, for large k, it tends asymptotically to the function $\omega = \sqrt{(kg)}$. Since the domain of existence of the localized waves is bounded by the frequencies $0 < \omega < N$, the asymptotic value is not achieved and, on the boundary of the domain of existence, there is a finite jump Δk between the point of the dispersion characteristic (10) and its asymptote.

It should be emphasized that the existence of localized waves is due to the fact that we have dispensed with the Boussinesq approximation (when an additional internal length scale Λ appears in the problem). Localized waves and travelling volume waves exist over the same frequency range.

Using the expressions for the volume energy density w and the energy flux density $\mathbf{J}$ [7], we can introduce a longitudinal energy density W and a frontal energy flux density S, which are natural in the case of localized states

$$W = \int_0^\infty w dz, \quad S = \int_0^\infty J_x dz$$

as well as the rate of energy transfer $\upsilon_e = S/W$. Here, we note that $J_z = 0$ in the case of the localized states being investigated, that is, the energy is strictly transferred in the direction of the x axis. Detailed calculations show that the rate of energy transfer is identical to the group velocity of the localized waves, that is, $\upsilon_e = \upsilon_g = \partial\omega(k)/\partial k$. In this case, the fluid particles in the localized wave move along closed elliptic trajectories where the ratio of the vertical and horizontal axes depends on z

$$\left|\frac{v_z}{v_x}\right| = \begin{cases} \operatorname{th} kz, & 0 < z < a \\ \operatorname{th} ka, & z > a \end{cases} \tag{11}$$

The vertical displacements on the bottom are equal to zero (the particles move horizontally) and these displacement increase as z increases, attaining a maximum value when $z = a$, after which the ratio of the axes of the displacement ellipse retain a constant value th ka above, in the stratified part of the fluid.

Treatment of the three-dimensional problem when there are no sources leads to its following formulation: it is required to find a function $u(x, y, z)$ which satisfies Laplace's equation when $0 < z < a$ and the equation

$$(1-\zeta^2)\Delta_1 u + \partial^2 u/\partial z^2 = \mu^2 u, \quad \Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \tag{12}$$

when $z > a$ with the boundary condition $u(x, y, 0) = 0$ and is bounded when $z \to \infty$ such that the function $\upsilon_z(x, y, z) = u(x, y, z)$ when $0 < z < a$ and, when $a < z < \infty$, $u(x, y, z)e^{\mu(z-a)}$ is continuously differentiable in the domain $z > 0$, $-\infty < x, y < +\infty$.

Since there is a distinguished direction (the z axis) which has been separated out in the problem, it is natural to separate the z and (x, y) variables. Assuming that $u = F(x, y)f(z)$, substituting this into Laplace's equation and Eq. (12) and separating the variables, we obtain that the functions $f(z)$ and $F(x, y)$ satisfy the equations

$$\begin{gathered} d^2 f/dz^2 + k_{z0}^2 f = 0, \quad 0 < z < a, \quad d^2 f/dz^2 + k_z^2 f = 0, \quad a < z < \infty \\ \Delta_1 F - k_{z0}^2 F = 0, \quad \Delta_1 F - (\mu^2 + k_z^2)/(1-\zeta^2) F = 0 \end{gathered} \tag{13}$$

where k_{z0}^2 and k_z^2 are separation constants for Laplace's equation and Eq. (12) respectively. The last two equations in (13) must be satisfied for any z. This condition specifies the relation between k_{z0}^2 and k_z^2

$$k_{z0}^2 = (\mu^2 + k_z^2)/(1-\zeta^2) \tag{14}$$

The general solutions of the first two equations in (13) have the form

$$f = c_{10}e^{ik_{z0}z} + c_{20}e^{-ik_{z0}z}, \quad 0 < z < a$$
$$f = c_1e^{ik_z z} + c_2e^{-ik_z z}, \quad a < z < \infty \tag{15}$$

Using the fact that u is equal to zero when $z = 0$ and the continuous differentiability of υ_z when $z = a$, we can find the relation between the coefficients c_{10}, c_{20}, c_1 and c_2. We obtain

$$c_{10} = -c_{20} = (c_1 + c_2)/(2i \sin k_{z0}a)$$
$$ik_{z0}(c_1 + c_2)\operatorname{ctg} k_{z0}a = (\mu + ik_z)c_1 + (\mu - ik_z)c_2 \tag{16}$$

The problem is now to elucidate the values which can be taken by k_z. In the general case, k_z is a complex number. If $\operatorname{Im} k_z \neq 0$, then one of the amplitudes, c_1 or c_2, vanish in order that the magnitude of u should not increase in an unbounded manner when $z \to \infty$. If, however, $\operatorname{Im} k_z = 0$, then each of the amplitudes c_1 and c_2 may be non-zero since, in this case, u does not increase without limit. We now consider these cases separately.

Suppose that $k_z = \lambda$, $\operatorname{Im} \lambda > 0$. In this case, we must have $c_2 = 0$ and u decreases exponentially with distance from the level $z = a$, which corresponds to localized states. Assuming that $k_{z0} = ik$ and using Eqs (14) and (16), we obtain relation (9).

Suppose that $k_z = p$, $\operatorname{Im} p = 0$. In this case $c_1, c_2 \neq 0$ and it is found from (16) that

$$\alpha(p) \equiv \frac{c_1}{c_2} = \frac{ik_{z0} - (\mu + ip)\operatorname{th} k_{z0}a}{ik_{z0} - (\mu - ip)\operatorname{th} k_{z0}a}$$

where k_{z0} is expressed in terms of p using Eq. (14). Here, the perturbation consists of a wave e^{-ipz} which is incident on the boundary $z = a$ and a wave $\alpha(p)e^{ipz}$ which is reflected from it so that this case corresponds to volume waves. Here, p is any positive number.

It follows from the foregoing treatment that, in fluid domains where there are no sources, the total field can be represented in the form of the superposition of the general solutions which have been found for the localized and volume waves

$$u(x, y, z) = F_s(x, y)\varphi_s(z) + \int_0^\infty F_v(x, y, p)\varphi_v(z, p)dp \tag{17}$$

where the functions $F_s(x, y)$ and $F_v(x, y, p)$ satisfy the equations

$$(\Delta_1 + k^2(\omega))F_s(x, y) = 0, \; (\Delta_1 + k_p^2(\omega))F_v(x, y, p) = 0 \tag{18}$$

The relation $k(\omega)$ is given by (10) and the longitudinal wave number of the volume waves k_p is given by the formula

$$k_p^2 = (p^2 + \mu^2)/(\zeta^2 - 1)$$

The specific form of the functions $F_s(x, y)$ and $F_v(x, y, p)$, which define the longitudinal structure of the localized and volume waves, is determined by the perturbation sources and the boundary conditions with respect to the horizontal coordinates, taking account of the formulation of the specific problem.

The functions $\varphi_s(z)$ and $\varphi_v(z, p)$, which describe the vertical structure of the localized and volume perturbations, have the form

$$\varphi_s(z) = \left[\frac{2i\lambda(\mu - i\lambda)\operatorname{sh} 2ka}{\mu \operatorname{sh} 2ka - 2i\lambda ka}\right]^{1/2} \begin{cases} \operatorname{sh} kz/\operatorname{sh} ka, & 0 < z < a \\ e^{i\lambda(z-a)}, & z > a \end{cases}$$

$$\varphi_v(z, p) = [2\alpha(p)]^{-1/2} \begin{cases} -\dfrac{2ip \operatorname{sh} k_p z}{k_p \operatorname{ch} k_p a - (\mu + ip)\operatorname{sh} k_p a}, & 0 < z < a \\ e^{-ip(z-a)} + \alpha(p)e^{ip(z-a)}, & z > a \end{cases} \tag{19}$$

$$\lambda = i(\mu - k \operatorname{cth} ka), \quad \alpha(p) = -\frac{k_p - (\mu - ip)\operatorname{th} k_p a}{k_p - (\mu + ip)\operatorname{th} k_p a}$$

They satisfy the orthogonality and completeness relations

$$\int_0^\infty h(z)\varphi_s^2(z)dz = 1, \quad \int_0^\infty h(z)\varphi_s(z)\varphi_\nu(z,p)dz = 0$$

$$\int_0^\infty h(z)\varphi_\nu(z,p)\varphi_\nu(z,q)dz = \delta(p-q); \quad p,q>0 \tag{20}$$

$$h(z)\left[\varphi_s(z)\varphi_s(z') + \int_0^\infty \varphi_\nu(z,p)\varphi_\nu(z',p)dp\right] = \delta(z-z')$$

with weighting function

$$h(z) = \begin{cases} (1-\zeta^2), & 0<z<a \\ 1, & z>a \end{cases}$$

The completeness relation (the last equality of (20)) proves the validity of representation (17) and the orthogonality relations (the first three equalities of (20)) enable one, using a known distribution $u(x,y,z)$, to separate out the localized and volume fields and, also, to find the amplitudes $F_s(x,y)$ and $F_\nu(x,y,p)$ using the formulae

$$F_s = \int_0^\infty h(z)u(x,y,z)\varphi_s(z)dz, \quad F_\nu = \int_0^\infty h(z)u(x,y,z)\varphi_\nu(z,p)dz \tag{21}$$

The orthogonality relations also enable one to solve the problem of the excitation of the field by a specified source distribution in a very convenient manner.

It is characteristic in the case of an ocean and internal reservoirs that the mixed layer is located above the stratified fluid and has a free surface. Localized waves also arise here and the dispersion of these waves is described by the implicit equation

$$\mu + \mu_*(k) = k\frac{kg\,\mathrm{ch}\,ka - \omega^2\,\mathrm{sh}\,ka}{\omega^2\,\mathrm{ch}\,ka - kg\,\mathrm{sh}\,ka} \tag{22}$$

For these waves to exist, it is not obligatory that the Boussinesq approximation be rejected. Unlike the localized bottom waves considered above, they only arise when $\omega > N$. In the case of a considerable thickness of the mixed layer, these waves are identical to the waves on the free surface of a homogeneous fluid.

The above analysis shows that, even in a model of an ideal medium, specific surface waves exist at the boundary of the layer of homogeneous fluid contiguous to the stratified layer, even when there is no discontinuity in the density at its boundary. It is also important that localized surface states arise over the same frequency rage ($\omega < N$) in which the volume waves are propagating, unlike the case of surface perturbations at a density discontinuity, which only exist when $\omega > N$.

Taking account of dissipative factors leads to a further complication of the flow pattern and to the appearance of additional boundary flows with characteristic localization length scales, which differ from the wave scales.

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